

Existence of Infinitely Many Solutions for a Quasilinear Elliptic Problem on Time Scales

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Abstract

We study a boundary-value quasilinear elliptic problem on a generic time scale. Making use of the fixed-point index theory, sufficient conditions are given to obtain existence, multiplicity, and infinite solvability of positive solutions.

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1 Introduction

We are interested in the study of the following quasilinear elliptic problem:

$$\begin{aligned} -(\phi_p(u^\Delta(t)))^\nabla &= f(u(t)) + h(t), \quad \forall t \in (0, T)_{\mathbb{T}} = \mathbb{T}, \\ u^\Delta(0) &= 0, \\ u(T) - u(\eta) &= 0, \quad 0 < \eta < T, \end{aligned} \tag{1}$$

where $\phi_p(\cdot)$ is the p -Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$ with q the Holder conjugate of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and \mathbb{T} is a time-scale. We assume the following hypotheses:

- (H1) The function $f : (0, T) \rightarrow \mathbb{R}^{+*}$ is a continuous function;
- (H2) The function $h : T \rightarrow \mathbb{R}^{+*}$ is left dense continuous (i.e. $h \in \mathbb{C}_{ld}(\mathbb{T}, \mathbb{R}^{+*})$). Here $\mathbb{C}_{ld}(\mathbb{T}, \mathbb{R}^{+*})$ denotes the set of all left dense continuous functions from $\mathbb{T} \rightarrow \mathbb{R}^+$; and $h \in L^\infty(0, T)$.

Results on existence of radially infinity many solutions for (1) are proved using (i) variational methods [5, 10], where solutions are obtained as critical points of some energy functional on a Sobolev space, by imposing appropriate conditions on f ; (ii) methods based on phase-plane analysis and the shooting method [11]; (iii) by adapting the technique of time maps [12]. For $p = 2$, $h = 0$, $\mathbb{T} = \mathbb{R}$, problem (1) becomes a boundary-value problem of differential equations. Our results extend and include results of the earlier works to the case of a generic time-scale \mathbb{T} , $p \neq 2$ and where h is not identically zero. In the case of $h = 0$, $p = 2$, many existence results of dynamic equations on time scales are available, using different fixed point theorems [6, 17]. We remark that there are not many results concerning the p -Laplacian problems on time scales [18]. In this paper we prove existence of solutions by constructing an operator whose fixed points are solutions of (1). Our main ingredient is the following well-known fixed-point theorem of index theory.

Theorem 1 ([13, 14]). *Suppose E is a real Banach space, and $K \subset E$ is a cone in X . Let $\Omega_r = \{u \in K, \|u\| < r\}$, and $F : \Omega_r \rightarrow K$ be a completely continuous operator satisfying $Fx \neq x$, for all $x \in \partial\Omega_r$. The following holds:*

- (i) if $\|Fx\| \leq \|x\|$, $\forall x \in \partial\Omega_r$, then $i(F, \Omega_r, K) = 1$;
- (ii) if $\|Fx\| \geq \|x\|$, $\forall x \in \partial\Omega_r$, then $i(F, \Omega_r, K) = 0$,

where i is the index of F .

The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing. The theory of time scales has been created in order to unify continuous and discrete analysis, allowing a simultaneous treatment of differential and difference equations, and to extend those theories to so-called delta/nabla-dynamic equations. A vast literature has already emerged in this field: see e.g. [2, 4, 7]. For an introduction to time scales with applications, we refer the reader to [8, 9].

The outline of the paper is as follows. In Section 2 we give some preliminary results with respect to the calculus on time scales. Section 3 is devoted to the existence of positive solutions using fixed-point index theory. The remaining sections deal with multiplicity and infinite solvability solutions for (1).

2 Preliminary results on time scales

We begin by recalling some basic concepts of time scales. Then, we prove some preliminary results that will be needed in the sequel.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the set \mathbb{R} of real numbers. The operators σ and ρ from \mathbb{T} to \mathbb{T} are defined in [15, 16]:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}.$$

They are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbb{T}$ is said to be left-dense, left-scattered, right-dense, or right-scattered, if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If \mathbb{T} has a right scattered minimum m , we define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise we set $\mathbb{T}_k = \mathbb{T}$. Similarly, if \mathbb{T} has a left scattered maximum M , we define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise we set $\mathbb{T}^k = \mathbb{T}$.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$ (assume t is not left-scattered if $t = \sup \mathbb{T}$). We define $f^\Delta(t)$ to be the number (provided it exists) such that given any $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t . We remark that f^Δ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference Δf (defined by $\Delta f(t) = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

Similarly, for $t \in \mathbb{T}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of f at the point t is defined in [7] to be the number $f^\nabla(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all } s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $f^\nabla(t) = f(t) - f(t-1)$ is the backward difference operator.

We say that a function f is left-dense continuous (*ld*-continuous for short), provided f is continuous at each left-dense point in \mathbb{T} and its right-sided limit exists at each right-dense point in \mathbb{T} . It is well-known that if f is *ld*-continuous and if $F^\nabla(t) = f(t)$, then one can define the nabla integral by

$$\int_a^b f(t) \nabla t = F(b) - F(a).$$

If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$

Function F is said to be an antiderivative of f . For more details on time scales, the reader can consult [1, 2, 3, 4, 8, 9] and references therein.

In the rest of the paper, \mathbb{T} is a closed subset of \mathbb{R} with $0 \in \mathbb{T}_k$, $T \in \mathbb{T}^k$. We denote $E = \mathbb{C}_{ld}([0, T], \mathbb{R})$, which is a Banach space with the maximum norm $\|u\| = \max_{[0, T]} |u(t)|$.

Lemma 2. Suppose that conditions (H1) and (H2) hold. Then, $u(t)$ is a solution of the boundary-value problem (1) if and only if $u(t) \in E$ is a solution of the following equation:

$$u(t) = \phi_q \left(\int_{\eta}^T (f(u(r) + h(r)) \nabla r) \right) + \int_0^t \phi_q \left(\int_s^T (f(u(r) + h(r)) \nabla r) \right) \Delta s. \quad (2)$$

Proof. By integrating the equation (1) on (s, T) , we have

$$\phi_p(u^\Delta(T)) = \phi_p(u^\Delta(s)) - \int_s^T (f(u(r) + h(r)) \nabla r).$$

Then,

$$\phi_p(u^\Delta(s)) = \phi_p(u^\Delta(T)) + \int_s^T (f(u(r) + h(r)) \nabla r).$$

Using the boundary conditions, we have

$$\phi_p(u^\Delta(s)) = \int_s^T (f(u(r) + h(r)) \nabla r).$$

Thus,

$$u^\Delta(s) = \phi_q \left(\int_s^T (f(u(r) + h(r)) \nabla r) \right).$$

Integrating the last equation on $(0, t)$, we have

$$\begin{aligned} u(t) &= u(0) + \int_0^t \phi_q \left(\int_s^T (f(u(r) + h(r)) \nabla r) \right) \Delta s \\ &= u^\Delta(\eta) + \int_0^t \phi_q \left(\int_s^T (f(u(r) + h(r)) \nabla r) \right) \Delta s \\ &= \phi_q \left(\int_{\eta}^T (f(u(r) + h(r)) \nabla r) \right) + \int_0^t \phi_q \left(\int_s^T (f(u(r) + h(r)) \nabla r) \right) \Delta s. \end{aligned}$$

Inversely, if we suppose that (2) holds, it is easy to get the first equation of (1) by derivation, and to see that u satisfies the boundary value conditions in (1). Furthermore, u is obviously positive since ϕ_q is non decreasing function and f and h are also positives functions. The proof of Lemma 2 is now complete. \square

On the other hand, we have $-(\phi_p(u^\Delta))^\nabla = f(u(t)) + h(t)$. Then, since $f, h \geq 0$, we have $(\phi_p(u^\Delta))^\nabla \leq 0$ and $(\phi_p(u^\Delta(t_2))) \leq (\phi_p(u^\Delta(t_1)))$ for any $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$. It follows that $u^\Delta(t_2) \leq u^\Delta(t_1)$ for $t_1 \leq t_2$. Hence,

$u^\Delta(t)$ is a decreasing function on $[0, T]$. This means that the graph of $u(t)$ is concave on $[0, T]$.

Let $K \subset E$ be the cone defined by

$$K = \{u \in E : u(t) \geq 0, u(t) \text{ is a concave function, } t \in [0, 1]\},$$

and $F : K \rightarrow E$ the operator

$$Fu(t) = \phi_q \left(\int_\eta^T (f(u(r)) + h(r)) \nabla r \right) + \int_0^t \phi_q \left(\int_s^T (f(u(r)) + h(r)) \nabla r \right) \Delta s.$$

It is easy to see that (1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator F . One can also verify that $F(K) \subset K$ and $F : K \rightarrow K$ is completely continuous.

3 Existence of positive solutions

We define two open subsets Ω_1 and Ω_2 of E :

$$\Omega_1 = \{u \in K : \|u\| < a\}, \quad \Omega_2 = \{u \in K : \|u\| < b\}.$$

Without loss of generality, we suppose that $b < a$. For convenience, we introduce the following notation:

$$A = \frac{a - \alpha \|h\|_\infty^{1/p-1}}{\alpha a}, \text{ where } \alpha = \phi_q(2^{p-2})\phi_q(T)(T+1), \\ B = \phi_p(T - \eta).$$

Theorem 3. *Besides (H1) and (H2), suppose that f also satisfies:*

- (i) $\max_{0 \leq u \leq a} f(u) \leq \phi_p(aA)$;
- (ii) $\min_{0 \leq u \leq b} f(u) \geq \phi_p(bB)$.

Then, (1) has a positive solution.

Proof. If $u \in \partial\Omega_1$, we have:

$$\begin{aligned} \|F(u)\| &\leq \phi_q \left(\int_\eta^T ((aA)^{p-1} + \|h\|_\infty) \nabla r \right) \\ &\quad + \int_0^T \phi_q \left(\int_s^T ((aA)^{p-1} + \|h\|_\infty) \nabla r \right) \Delta s \\ &\leq \phi_q \left(((aA)^{p-1} + \|h\|_\infty)(T - \eta) \right) \\ &\quad + \int_0^T \phi_q \left(((aA)^{p-1} + \|h\|_\infty)(T - s) \right) \Delta s \end{aligned}$$

$$\begin{aligned}
&\leq \phi_q((aA)^{p-1} + \|h\|_\infty) \phi_q(T) \\
&\quad + \phi_q((aA)^{p-1} + \|h\|_\infty) \int_0^T \phi_q(T-s) \Delta s \\
&\leq \phi_q \left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1} \right) \phi_q(T) \\
&\quad + \phi_q \left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1} \right) \phi_q(T) T \\
&\leq \phi_q \left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1} \right) \phi_q(T)(T+1).
\end{aligned}$$

Using the elementary inequality

$$x^p + y^p \leq 2^{p-1}(x+y)^p,$$

and the form of A , it follows that

$$\begin{aligned}
\|F(u)\| &\leq \phi_q(T+1)(2^{p-2})(aA + \|h\|_\infty^{1/p-1}) \\
&\leq \|u\| = a.
\end{aligned}$$

Therefore, $\|Fu\| \leq \|u\|$ for all $u \in \partial\Omega_1$. Then, by Theorem 1,

$$i(F, \Omega_1, K) = 1. \tag{3}$$

On the other hand, for $u \in \partial\Omega_2$ we have:

$$\begin{aligned}
\|F(u)\| &\geq \phi_q \left(\int_\eta^T f(u(r)) \nabla r \right) + \int_0^t \phi_q \left(\int_s^T (f(u(r))) \nabla r \right) \Delta s. \\
&\geq Bb\phi_q(T-\eta) \\
&\geq b = \|u\| \text{ (since } B = \phi_p(T-\eta)).
\end{aligned}$$

Therefore, $\|Fu\| \geq \|u\|$ for all $u \in \partial\Omega_2$. By Theorem 1,

$$i(F, \Omega_2, K) = 0. \tag{4}$$

It follows by (3) and (4) that

$$i(F, \Omega_1 \setminus \overline{\Omega_2}, K) = 1.$$

Then T has a fixed point $u \in \Omega_1 \setminus \overline{\Omega_2}$. Obviously, u is a positive solution of problem (1) and $b < \|u\| < a$. The proof of Theorem 3 is complete. \square

4 Multiplicity

By multiplicity we mean the existence of an arbitrary number of solutions. We now obtain results on the multiplicity of positive solutions for (1) under the following assumptions: we suppose that there exist positive real numbers $0 < a_1 < a_2 < \dots < a_{k+1}$, such that

(i) $\max_{0 \leq u \leq a_{2i-1}} f(u) \leq \phi_p(a_{2i-1}A), i = 1, \dots, [\frac{k+2}{2}];$

(ii) $\min_{0 \leq u \leq a_{2i}} f(u) \geq \phi_p(a_{2i}B), i = 1, \dots, [\frac{k+1}{2}];$

where $[n]$ denote the integer part of n .

Theorem 4. Assume that (i)-(ii) hold. Then, problem (1) has at least k positive solutions u_1, \dots, u_k such that

$$a_i < \|u_i\| < a_{i+1}, \quad i = 1, \dots, k.$$

Proof. By continuity, there exist

$$0 < b_1 < a_1 < c_1 < b_2 < a_2 < c_2 < \dots < c_k < b_{k+1} < a_{k+1} < +\infty$$

such that

$$\min_{0 \leq u \leq b_{2i-1}} f(u) \geq \phi_p(b_{2i-1}B), \quad \min_{0 \leq u \leq c_{2i-1}} f(u) \geq \phi_p(c_{2i-1}B),$$

for $i = 1, \dots, [\frac{k+2}{2}]$, and

$$\max_{0 \leq u \leq c_{2i}} f(u) \leq \phi_p(c_{2i}A), \quad \max_{0 \leq u \leq b_{2i}} f(u) \leq \phi_p(b_{2i}A),$$

for $i = 1, \dots, [\frac{k+1}{2}]$. Then, calling Theorem 3 to each interval $(c_i, b_{i+1}), i = 1, \dots, k$, we obtain the intended result. \square

5 Infinite solvability

Theorem 5. If the following two conditions hold,

$$(i) \liminf_{a \rightarrow 0} \frac{\max_{0 \leq u \leq a} \{f(u)\}}{a^{p-1}} \leq \phi_p(A),$$

$$(ii) \limsup_{b \rightarrow 0} \frac{\max_{0 \leq u \leq b} \{f(u)\}}{b^{p-1}} \geq \phi_p(B),$$

then, problem (1) has a sequence of positive solutions $(u_k)_{k \geq 1}$ such that $\|u_k\| \rightarrow 0$ as $k \rightarrow \infty$.

Proof. From (i) and (ii), there exists a sequence of pairs of positive numbers $(a_k, b_k)_{k \geq 1}$ convergent to $(0, 0)$ such that

$$\max_{0 \leq u \leq a_k} f(u) \leq \phi_p(a_k A),$$

$$\min_{0 \leq u \leq b_k} f(u) \geq \phi_p(b_k B).$$

Suppose that

$$a_1 > b_1 > a_2 > b_2 > \dots > a_k > b_k > \dots$$

Calling Theorem 3 on each pair $(a_k, b_k)_{k \geq 1}$, we conclude that (1) has a sequence of positive solutions $(u_k)_{k \geq 1}$ such that $b_k \leq \|u_k\| \leq a_k$. \square

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